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THE METHOD OF DUAL SERIES IN TERMS OF BESSEL FUNCTIONS IN MIXED PROBLEMS OF THE THEORY OF ELASTICITY FOR A CIRCULAR PLATE

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Naturally mixed axisymmetric problems of the theory of elasticity are studied for the case of a circular plate of finite radius and thickness. Mixed boundary conditions are specified at the plate faces. The boundary conditions given at the side surface are chosen so as to make the problems reducible to dual series in terms of Bessel functions. The dual series are solved using the general method given in [1] and based on reducing the series to an infinite algebraic system of the first kind with a singular matrix of coefficients. Exact inversion of the principal singular part yields an infinite system of the second kind, and this is solved in an approximate manner by reduction on a digital computer. In addition, the principal term of the asymptotics of the solution is obtained for the case of a plate of relatively small thickness. Two problems of torsion of a stamp on a plate are studied in detail as examples, and results of the numerical computations are given.

1. Reduction of the dual series to an infinite system. Using the cvlindrical coordinate system, let us consider an elastic plate occupying the region $0 \leqslant z < h$, $r \leqslant R$. The contact problem of impressing a stamp into the face $z = h$ of the plate rigidly clamped by its base $z = 0$, or lying freely on a rigid support $z = 0$, under the condition that on the side surface we either have

$$u_r(z, R) = 0, \quad \tau_{rz}(z, R) = 0$$

or

$$\sigma_r(z, R) = 0, \quad u_z(z, R) = 0$$

can be reduced to the problem of investigating a dual series equation of the form

$$\sum_{k=1}^{\infty} a_k K(u_k) J_\nu(u_k x) = f(x) \quad (0 \leqslant x \leqslant a) \tag{1.1}$$

$$\sum_{k=1}^{\infty} a_k J_\nu(u_k x) = 0 \quad (a < x \leqslant R)$$

$$K(u) = A \frac{p_1(u^2)}{p_2(u^2)} = A \prod_{n=1}^{\infty} \left(1 + \frac{u^2}{\delta_n^2}\right)\left(1 + \frac{u^2}{\gamma_n^2}\right)^{-1}, \quad A = \text{const} \tag{1.2}$$

Here a_k are unknown coefficients, $J_\nu(x)$ are the Bessel functions ($\nu \geqslant -1/2$) and $K(u)$ is an even meromorphic function which can be written in the form (1.2), where $\pm i\delta_n$ and $\pm i\gamma_n$ denote denumerable sets of simple zeros and poles. When considering the equation (1.1) in its general form, we shall assume that δ_n and γ_n increase monotonously in modulo with their index, thus ensuring the convergence of the infinite product (1.2). However, the following estimate holds on any correct system of contours C_n in the complex variable u plane:

$$K(u) = O(|u|^p), \quad p \leqslant 0, \ n \to \infty$$

and u_k are zeros of the equation

$$\left[\frac{d}{dx} J_\nu(u_k x) + \alpha J_\nu(u_k x)\right]_{x=R} = 0 \tag{1.3}$$

Let $f(x) = J_\nu(i\varepsilon x)$ in (1.1) and remember that the function $f(x)$ can in general be expanded into a series in terms of the functions $J_\nu(i\varepsilon_k x)$. Then the dual series equation (1.1) will be equivalent to the following infinite system:

$$(B_1 + B_2)\, X(\varepsilon) = D_1(\varepsilon) + D_2(\varepsilon) \tag{1.4}$$

$$B_1 = \left\{\frac{\gamma_m K_{\nu-1}(\gamma_m a)\, I_\nu(\delta_n a) + \delta_n I_{\nu-1}(\delta_n a)\, K_\nu(\gamma_m a)}{(\delta_n^2 - \gamma_m^2)\, K_\nu(\gamma_m a)\, I_\nu(\delta_n a)}\right\}$$

$$B_2 = -\left\{\frac{\gamma_m I_{\nu-1}(\gamma_m a)\, I_\nu(\delta_n a) - \delta_n I_{\nu-1}(\delta_n a)\, I_\nu(\gamma_m a)}{(\delta_n^2 - \gamma_m^2)\, K_\nu(\gamma_m a)\, I_\nu(\delta_n a)} B(\gamma_m, R)\right\}$$

$$D_1(\varepsilon) = \left\{\frac{\gamma_m K_{\nu-1}(\gamma^m a)\, I_\nu(\varepsilon a) + \varepsilon I_{\nu-1}(\varepsilon a)\, K_\nu(\gamma_m a)}{(\gamma_m^2 - \varepsilon^2)\, K_\nu(\gamma_m a)\, K(i\varepsilon)}\right\}$$

$$D_2(\varepsilon) = -\left\{\frac{\gamma_m I_{\nu-1}(\gamma_m a)\, I_\nu(\varepsilon a) - \varepsilon I_{\nu-1}(\varepsilon a)\, I_\nu(\gamma_m a)}{(\gamma_m^2 - \varepsilon^2)\, K_\nu(\gamma_m a)\, K(i\varepsilon)} B(\gamma_m, R)\right\}$$

$$B(\gamma_m, R) = \frac{\gamma_m K_{\nu-1}(\gamma_m R) - (\alpha - \nu/R)\, K_\nu(\gamma_m R)}{\gamma_m I_{\nu-1}(\gamma_m R) + (\alpha - \nu/R)\, I_\nu(\gamma_m R)}$$

Here $B_i = \{b_{m,n}^{(i)}\}$ denote the matrices and $D_i(\varepsilon) = \{d_m^{(i)}(\varepsilon)\}$, $X(\varepsilon) = \{x_n(\varepsilon)\}$ the column vectors of infinite order $(i = 1, 2)$. The coefficients a_k ($k = 1, 2, \ldots$) of (1.1) are connected with the solution of the infinite system (1.4) by the following relations:

$$q(x) = \begin{cases} K^{-1}(i\varepsilon)\, J_\nu(i\varepsilon x) + \sum\limits_{n=1}^{\infty} x_n(\varepsilon)\, J_\nu(i\delta_n x)\, I_\nu^{-1}(\delta_n a) & (0 \leqslant x \leqslant a) \\ 0 & (a < x \leqslant R) \end{cases} \tag{1.5}$$

$$q(x) = \sum_{k=1}^{\infty} a_k J_\nu(u_k x) \tag{1.6}$$

Assuming that $|\gamma_m|$, $|\delta_m| \sim m$ $(\delta_m \neq \gamma_m)$ as $m \to \infty$ and taking into account the asymptotic representation of the functions $K_\nu(x)$ and $I_\nu(x)$ with $x \to \infty$, we can regularize the system (1.4) by separating and inverting an infinite matrix A with the elements

$$a_{mn} = (\delta_n - \gamma_m)^{-1}, \; A = \{a_{mn}\} \tag{1.7}$$

Using the matrix A^{-1} which is an inverse of A, with the elements [2]

$$\tau_{nm} = [(K_-^{-1}(i\gamma_m))' K_+'(-i\delta_n)(\gamma_m - \delta_n)]^{-1}, A^{-1} = \{\tau_{nm}\} \tag{1.8}$$

$$K_+(u) = K_-(-u) = \sqrt{A} \prod_{n=1}^{\infty} \left(1 + \frac{u}{i\delta_n}\right)\left(1 + \frac{u}{i\gamma_n}\right)^{-1}$$

we can reduce (1.4) to an infinite system of the second kind

$$X(\varepsilon) = A^{-1}(D_1(\varepsilon) + D_2(\varepsilon)) - A^{-1}(B_1 + B_2 - A) X(\varepsilon) \tag{1.9}$$

The above system can be solved for specified values of the parameters appearing in it, using the method of consecutive approximation [2].

Passing in the infinite system (1.4) to the limit as $R \to \infty$, the matrix B_2 and the column vector $D_2(\varepsilon)$ both vanish and we obtain the infinite system

$$B_1 X(\varepsilon) = D_1(\varepsilon)$$

corresponding to the contact problem for an elastic layer.

2. Torsion of a stamp on a circular plate. Let us consider the solutions of two mixed problems of the theory of elasticity concerning torsion of a stamp on a circular plate with its side edge rigidly clamped (Problem 1) or with its side edge stress-free (Problem 2). The above problems will be equivalent to the following two boundary value problems written in terms of the displacement function $v(r, z)$ along the θ axis

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{2.1}$$

$$v = \delta r \quad \text{for } 0 \leqslant r \leqslant a, \; z = h \tag{2.2}$$

$$\tau_{z\varphi} = G\frac{\partial v}{\partial z} = 0 \quad \text{for } a < r < R, \; z = h$$

$$v = 0 \quad \text{for } 0 \leqslant r \leqslant R, \; z = 0$$

$$v = 0 \quad \text{for } r = R, \; 0 \leqslant z \leqslant h \text{ (problem 1)} \tag{2.3}$$

$$\tau_{r\varphi} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = 0 \quad \text{for } r = R, \; 0 \leqslant z \leqslant h \text{ (problem 2)} \tag{2.4}$$

Here δ is the angle of rotation of the stamp, a is the stamp radius, R is the plate radius, h is the plate height and G is the shear modulus.

Seeking $v(r, z)$ in the form

$$v(r,z) = \delta \sum_{n=1}^{\infty} a_n \operatorname{sh} u_n z (u_n \operatorname{ch} u_n h)^{-1} J_1(u_n r) \tag{2.5}$$

we find that the coefficients a_n are given by the dual series equation (1.1) in which we set

$$\nu = 1,\ f(x) = x,\ K(u) = u^{-1} \operatorname{th} uh \tag{2.6}$$

The differential equation (2.1) and the boundary conditions (2.2) will be satisfied here. Complying now with the conditions (2.3) and (2.4), we obtain the following equations defining the constants u_k for the Problems 1 and 2, respectively:

$$J_1(u_k R) = 0, \quad J_2(u_k R) = 0 \tag{2.7}$$

The above equations represent a particular case of (1.3) for $\alpha = \infty$ and $\alpha = 1$.

The contact stresses under the stamp are given by the relation

$$\tau_{r\varphi}(r, 0) = G\delta \sum_{n=1}^{\infty} a_n J_1(u_n r) \quad (0 \leqslant r \leqslant a) \tag{2.8}$$

Taking into account the relation

$$r = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} J_1(\varepsilon r)$$

and (1.5), (1.6), we obtain

$$\tau_{r\varphi}(r, 0) = G\delta \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} q(x) = \tag{2.9}$$

$$G\delta \left(\frac{r}{h} + 2 \sum_{n=1}^{\infty} y_n I_1(\delta_n r) I_1^{-1}(\delta_n a) \right), \quad y_n = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} x_n(\varepsilon)$$

where $x_n(\varepsilon)$ is the solution of the infinite system (1.4) for $\nu = 1$ and $\alpha = \infty$ (Problem 1) and for $\alpha = 0$ (Problem 2). Differentiating (1.4) with respect to ε and setting $\varepsilon = 0$, we obtain the following infinite system for determining the coefficients y_n $(n = 1, 2, \ldots)$

$$B \cdot Y = D,\ B = B_1 + B_2 = \{b_{mn}\},\ Y = \{y_n\},\ D = \{d_n\} \tag{2.10}$$

$$d_m = \frac{a\,[K_2(\gamma_m a) - B(\gamma_m, R) I_2(\gamma_m a)]}{2h\gamma_m K_1(\gamma_m a)}, \quad \delta_n = \frac{\pi n}{h}, \quad \gamma_m = \frac{\pi(2m-1)}{2h}$$

$$B(\gamma_m, R) = -\frac{K_1(\gamma_m R)}{I_1(\gamma_m R)} \quad \text{Problem 1}$$

$$B(\gamma_m, R) = \frac{K_2(\gamma_m R)}{I_2(\gamma_m R)} \quad \text{Problem 2}$$

The matrices B_1 and B_2 are given by (1.4) with $\nu = 1$.

Using (2.9), we obtain the following relation connecting the moment M applied to the stamp with the angle of rotation δ

$$M = G\delta a^3\pi\left[\frac{a}{2h} + \frac{4}{a}\sum_{n=1}^{\infty} y_n \frac{I_2(\delta_n a)}{\delta_n I_1(\delta_n a)}\right] \tag{2.11}$$

For the Problem 2 with $R = a$ (2.10) yields $d_m = 0,\ b_{mn} \neq 0,$ consequently, the solution of the infinite system (2.10) is $y_n \equiv 0.$ Thus when $R = a$ the formulas (2.9) and (2.11) yield the exact solution of the problem of torsion of a rod by a stamp of the same diameter.

Let us find the principal term of the asymptotics of the solution of (2.1) for small h / a and $(R - a) / h > 0$. Using the relation $\delta_n = \pi n / h,\ \gamma_m = \pi (2m - 1) / 2h$ and the asymptotics fo the functions $I_n(x)$ and $K_n(x)$ for large values of the argument x, we have the following expressions for the small values of h / a

$$B_1 = A + B_3, \quad B_3 \sim \{3\,[8a^2\gamma_m\delta_n(\delta_n - \gamma_m)]^{-1}\} \tag{2.12}$$

$$D \sim \left\{a\,\frac{1 \pm \exp(-2\gamma_m(R-a))}{2h\gamma_m}\right\} = D_0$$

$$B_2 \sim \left\{\frac{\pm\exp(-2\gamma_m(R-a))}{\delta_n + \gamma_m}\right\}$$

where the elements of the matrix A are given by (1.7). Here and henceforth the plus sign refers to the Problem 1 and the minus sign to Problem 2.

Using the fact that the elements of the matrices B_2 and B_3 are small compared to those of the matrix A we obtain, for $h / a \to 0$ and $(R - a) / h > R_0 > 0,$ the infinite system

$$AY_0 = D_0, \quad Y_0 = \{y_n^\circ\} \tag{2.13}$$

the solution of which will represent the principal term of the asymptotic to the solution of (2.10) for small h / a and $(R - a) / h > R_0 > 0.$ The matrix A has an inverse A^{-1} with elements (1.8) which, for the problems in question, assume the form

$$\tau_{nm} = \frac{(2n-1)!!\,(2m-3)!!}{h\,(2n-2)!!\,(2m-2)!!\,(2n-2m+1)} \tag{2.14}$$

Moreover, the solution of the inifinite system (2.13) with its right-hand side equal to $\{a / (2h\gamma_m)\}$. is known [2]. Using this fact and the inverse matrix (2.14), we obtain the solution of (2.13) in the form

$$y_n^\circ = \frac{a\,(2n-1)!!}{h\,(2n-2)!!}\left[\frac{1}{4\pi} \pm \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{(2m-3)!!\exp(-\pi(R-a)(2m-1)/h)}{(2m-2)!!\,(2n-2m+1)\,(2m+1)}\right] \tag{2.15}$$

For small h / a the formula (2.11) can be transformed asymptotically to the form

$$M = \pi G\delta a^3 \left(\frac{a}{2h} + \frac{4h}{\pi a} \sum_{n=1}^{\infty} n^{-1} y_n^\circ\right) \qquad (2.16)$$

and the formula (2.9), for $0 < \varepsilon_0 < r \leqslant a$, to the form

$$\tau_{r\varphi}(r, 0) = G\delta\left[\frac{r}{h} + 2\sqrt{\frac{a}{r}} \sum_{n=1}^{\infty} y_n^\circ E^n(r)\right] \qquad (2.17)$$

$$E(r) = \exp\left[-\frac{\pi}{h}(a - r)\right]$$

If $r \to 0$, then

$$\tau_{r\varphi}(r, 0) = G\delta\left[\frac{r}{h} + \frac{\pi^2 r}{h}\sqrt{\frac{2a}{h}} \sum_{n=1}^{\infty} y_n^\circ n\sqrt{n} E^n(0) + O(r^3)\right] \qquad (2.18)$$

The series in (2.17) converges uniformly for all $r \leqslant a - \varepsilon_1$, $\varepsilon_1 > 0$. When $r \to a$, the series diverges and this implies that the function $\tau_{r\varphi}(r, 0)$ has a singularity. We transform (2.15) to the form

$$y_n^\circ = \frac{a}{2h} \frac{(2n-1)!!}{(2n)!!}\left[1 \pm \frac{2}{\pi}\gamma_0\left(\frac{R-a}{h}\right) \mp \frac{2}{\pi}\gamma_n\left(\frac{R-a}{h}\right)\right] \qquad (2.19)$$

$$\gamma_n(x) = -\sum_{m=1}^{\infty} \frac{(2m-3)!! \exp[-\pi x(2m-1)]}{(2m-2)!!(2n-2m+1)} \qquad (n = 0, 1, \ldots)$$

Substituting (2.19) into (2.17) and summing on of the series, we obtain

$$\tau_{r\varphi}(r, 0) = G\delta\left\{\frac{r}{h} + \sqrt{\frac{a}{r}}\frac{a}{h}\left[1 \pm \frac{2}{\pi}\gamma_0\left(\frac{R-a}{h}\right)\right] \times \right. \qquad (2.20)$$

$$[(1 - E(r))^{-1/2} - 1] \mp \frac{2a}{\pi h}\sqrt{\frac{a}{r}} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \times$$

$$\left.\gamma_n\left(\frac{R-a}{h}\right) E^n(r)\right\} \qquad (0 < \varepsilon_0 < r \leqslant a)$$

The above series converges when $r = a$ and the singularity is shown explicitly. For large values of the parameter $(R - a) / h$ we have

$$y_n^\circ = \frac{a}{2h} \frac{(2n-1)!!}{(2n)!!}\left[1 \pm \frac{4}{\pi} E^{-1}(R) \frac{n}{2n-1}\right] \qquad (2.21)$$

and hence for a small, fixed value of the parameter h / a and large values of the parameter $(R - a) / h$ we obtain

$$\tau_{r\varphi}(r,0) = G\delta\left\{\frac{r}{h} + \frac{a}{h}\sqrt{\frac{a}{r}}\left[(1-E(r))^{-1/2} - 1 \pm \pm\frac{2}{\pi}\frac{E(r)}{E(R)}(1-E(r))^{-1/2}\right]\right\} \qquad (2.22)$$

$$M = \pi G\delta a^3\left[\frac{a}{2h} + \frac{2\ln 4}{\pi} \pm \frac{8}{\pi^2 E(R)}\right]$$

We note that for small h/a and for $(R-a)/h \to \infty$ the formulas (2.22) will coincide with the corresponding results of [3] concerned with the problem of torsion of a layer by a stamp.

The formula (2.22) enables us to conclude that for a small, fixed value of h/a the influence of the side surface of the plate on the distribution of the contact stresses under the stamp decays exponentially with increasing value of the parameter $(R-a)/h$ already becoming insignificant when $(R-a)/h \geqslant 1$, this influence can be neglected. Numerical computations show that the formula (2.22) can be used with a relative error not exceeding 10% when $h/a \leqslant 0.5$ and $(R-a)/h \geqslant 0.3$.

To obtain the solutions of the problems in question for the domain of variation of parameters in which the asymptotic formulas derived above become ineffective, we shall truncate the infinite system (2.10) having previously regularized it in accordance with the shceme (1.9). We write the truncated system in the form

$$y_n = \sum_{m=1}^{\infty}\tau_{nm}d_m^{\cdot} + \sum_{k=1}^{N}\left(\sum_{m=1}^{N}\tau_{nm}b_{mk}\right)y_k \quad (n=1,2,\ldots,N) \qquad (2.23)$$

and find its solution using a digital computer.

The value of N is chosen according to the accuracy needed. The numerical computations show that the convergence of the truncation method improves with increasing parameter $(R-a)/h$. In the course of solving (2.23), we have also obtained the value of the quantity

$$\mu = \max_{1\leqslant n\leqslant N}\sum_{k=1}^{N}\left|\sum_{m=1}^{N}\tau_{nm}b_{mk}^{\cdot}\right|$$

and this enabled us to make an approximate estimate of the values of R/a and h/a for which the infinite system becomes completely regular.

Thus e.g. for $N=43$ and $\mu<1$ if $R/a = 1.05$, $h/a \leqslant 4.10$ (0.28), $R/a = 1.1$, $h/a \leqslant 4.05$ (0.39); $R/a=2$, $h/a \leqslant 4.28$ (2.09); $R/a=4$, $h/a \leqslant 4.3$ (4.07). Here the number within the bracket corresponds to Problem 2, and the number preceding the bracket corresponds to Problem 1. Moreover, increasing the last digit in the inequalities by one produces the least value of the parameter h/a for which $\mu \geqslant 1$.

Figure 1 depicts the dependence of the quantity $\tau = \tau_{r\varphi}(r,0)(G\delta)^{-1}$ on r/a for some values of the parameters h/a and $(R-a)/a$ for Problem 1, computed by truncating the infinite system according to the scheme (2.23) and formula (2.9), with $N=43$. Curves 1–7 in Fig. 1a are constructed for the following values of the parameters h/a and $(R-a)/a$: 1, ∞; 1, 0.5; 1, 0.1; 0.3, ∞; 0.3, 0.1; 0.3, 0.05; 0.3, 0.01. and the curves 1–6 in Fig. 1b, for 2, ∞; 2, 0.5; 0.5,

∞; 0.5, 0.1, 0.5, 0.05; 0.5, 0.01. Fig. 2 depicts the dependence of the quantity $m = 3M(16\ G\delta a^3)^{-1}$ on h/a for $(R-a)/a = \infty$, 0.1, 0.05, 0.01 (the corresponding curves are 1-4). For $(R-a)/a = \infty$ the curves are constructed according to the asymptotic formulas of [3] where the problem of torsion of a layer by a stamp was investigated. The latter curves practically merge with the curves in Fig. 1 depicting the stress distribution under the stamp, provided that $(R-a)/h \gg 1$.

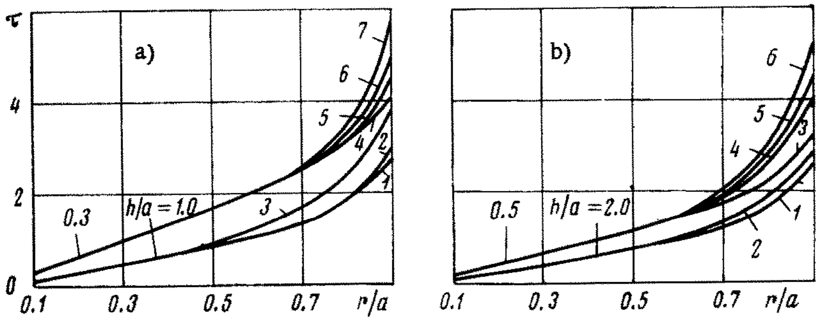

Fig. 1

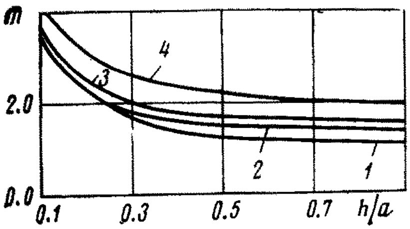

Fig. 2

The graphs show that the side surface of the plate exerts a predominant influence on the stress distribution under the stamp, near its boundary.

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